Math 53, Discussion 107

Monday, April 27, 2020 9:33 AM

16.8 (Stokes' theorem)
16.9 (Divergence theorem)

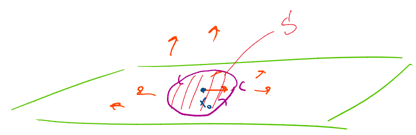
Theorem (Stokes)

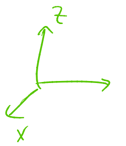
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

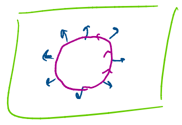
C positively oriented closed curve
→ S surface with boundary S

Interpret to give a meaning of curl

Given a vector field $F:\mathbb{R}^3\to\mathbb{R}^3$, to find $\text{Curl}(F)\big|_{X_0}$ in z direction:







Compute line integral of a small loop in x-y plane

$$\oint_C F\cdot dr \overset{\text{stokes}}{=} \iint_S (\nabla\times F)\cdot dS = \iint_S \underbrace{(\nabla\times F)\cdot\overset{\text{normal}}{\langle 0,0,1\rangle}}_{z\text{ component of curl of }F}\, dS$$

$$= \text{Area}(S)\cdot(\text{Avg curl of }F\text{ in z direction})$$

Theorem (Divergence theorem)

$$\underbrace{\iint_S \overset{\text{vector field}}{\vec{F}}\cdot \overset{\text{surface integral}}{d\vec{S}}}_{\text{Flux of F through S}} = \underbrace{\iiint_V \underbrace{\nabla\cdot\vec{F}}_{\text{divergence of vector field}}\, \overset{\text{volume}}{dV}}_{\text{total divergence of F in a 3-D Region}}$$

S: closed surface

V: 3-D region enclosed by S

Given vector field F, if S is a small sphere with center X_0

=> RHS of divergence theorem = flux of F through sphere

LHS = (Volume of sphere) (Average divergence)

So if $\nabla\cdot F(X_0)>0$ => have a source (stuff going out)

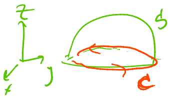
$\nabla\cdot F(X_0)<0$ => have a sink (stuff going in)

Exercises:

1) Compute $\iint_S \nabla\times \boldsymbol{F}\cdot d\boldsymbol{S}$ for $F = \langle 2y\cos z, e^x\sin z, xe^y\rangle$, on the upper half of the sphere centered at the origin with radius 9
2) Compute $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ for $F = \langle yz, 2xz, e^{xy}\rangle$ C the circle $x^2+y^2=16, z=5$
1) Compute $\iint_S F\cdot dS$ for $F = \langle exy^2, xe^z, z^3\rangle$ over the surface bounded by $y^2+z^2=1$ and the $x = -1, x = 2$
   (a) Are the points P_1 and P_2 sources or sinks for the vector field **F** shown in the figure? Give an explanation based

1) $\iint_S \nabla\times F\cdot dS \overset{\text{stokes}}{=} \oint_C F\cdot dS$

param. C as $r = (9\cos\theta, 9\sin\theta, 0)$



$$= \oint_C P\,dx + \overset{0}{\cancel{Q\,dy}} + \overset{0}{\cancel{R\,dz}}$$

$dx = -9\sin\theta$

$dy = 9\cos\theta$

$dz = 0$

$Q(r(t)) = e^{9\cos\theta}\sin 0 = 0$

$$= \int_0^{2\pi} 2(9\sin\theta)\cos(0)(-9\sin\theta)\,d\theta$$

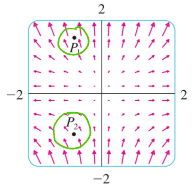
the surface bounded by $y^2 + z^2 = 1$ and the $x = -1, x = 2$

(a) Are the points P_1 and P_2 sources or sinks for the vector field **F** shown in the figure? Give an explanation based solely on the picture.

(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).

$$= \int_0^{2\pi} z\, \underset{y}{(9\sin\theta)}\, \underset{\overset{\uparrow}{z}}{\cos(0)}\, \underset{\overset{\uparrow}{dx}}{(-9\sin\theta)}\, d\theta$$

2)



Source at P_1

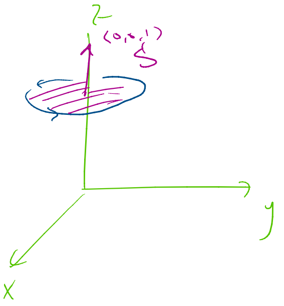
$\nabla \cdot F(P_1) > 0$

Sink at P_2

$\nabla \cdot F < 0$

2) $\oint_C F \cdot dr \quad \iint_S \nabla \times F \cdot d\vec{S}$

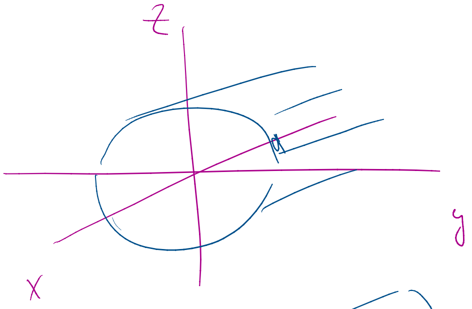
$$\nabla \times F = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} yz \\ 2xz \\ e^{xy} \end{pmatrix} = \begin{bmatrix} xe^{xy} - 2x \\ -ye^{xy} + y \\ 2z - z \end{bmatrix}$$



$$\nabla \times F \cdot d\vec{S} = (\nabla \times F) \cdot \underset{\overset{\uparrow}{\vec{n}}}{(0,0,1)}\, dS = z\, dS$$

$\iint_S z\, dS$ z is constant 5 over S, so can pull out

$= 5 \text{ Area}(S) = (5)(16\pi) = 80\pi$



1) Compute $\iint_S F \cdot dS$ for $F = \langle \overset{e^x}{e\cancel{x}}y^2, xe^z, z^3 \rangle$ over the surface bounded by $y^2 + z^2 = 1$ and the $x = -1, x = 2$

$$\iint F \cdot dS = \iiint_V \nabla \cdot F\, dV$$

$$\nabla \cdot F = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{bmatrix} e^x y^2 \\ xe^z \\ z^3 \end{bmatrix}$$

$$= e^x y^2 + 0 + 3z^2$$